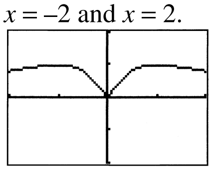
Chapter 5

Applications of Derivatives

Section 5.1 Extreme Values of Functions (pp. 191–199)

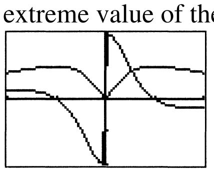
Exploration 1 Finding Extreme Values

1. From the graph we can see that there are three critical points: $x = -1, 0, 1$.
   Critical point values:
   $f(-1) = 0.5,\ f(0) = 0,\ f(1) = 0.5$ Endpoint values: $f(-2) = 0.4,\ f(2) = 0.4$
   Thus f has absolute maximum value of 0.5 at $x = -1$ and $x = 1$, absolute minimum value of 0 at $x = 0$, and local minimum value of 0.4 at $x = -2$ and $x = 2$.



[−2, 2] by [−1, 1]

2. The graph of f' has zeros at $x = -1$ and $x = 1$ where the graph of f has local extreme values. The graph of f' is not defined at $x = 0$, another extreme value of the graph of f.



[−2, 2] by [−1, 1]

3. We can write $f(x) = \begin{cases} \dfrac{-x}{x^2+1} & \text{for } x < 0 \\ \dfrac{x}{x^2+1} & \text{for } x \ge 0 \end{cases}$,
   so the Quotient Rule gives
   $$f'(x) = \begin{cases} -\dfrac{1-x^2}{(x^2+1)^2} & \text{for } x < 0 \\ \dfrac{1-x^2}{(x^2+1)^2} & \text{for } x \ge 0 \end{cases},$$
   which can be written as $f'(x) = \dfrac{|x|}{x} \cdot \dfrac{1-x^2}{(x^2+1)^2}$.

Quick Review 5.1

1. $f'(x) = \dfrac{1}{2\sqrt{4-x}} \cdot \dfrac{d}{dx}(4-x) = \dfrac{-1}{2\sqrt{4-x}}$

2. $f'(x) = \dfrac{d}{dx} 2(9-x^2)^{-1/2}$
   $= -(9-x^2)^{-3/2} \cdot \dfrac{d}{dx}(9-x^2)$
   $= -(9-x^2)^{-3/2}(-2x)$
   $= \dfrac{2x}{(9-x^2)^{3/2}}$

3. $g'(x) = -\sin(\ln x) \cdot \dfrac{d}{dx} \ln x = -\dfrac{\sin(\ln x)}{x}$

4. $h'(x) = e^{2x} \cdot \dfrac{d}{dx} 2x = 2e^{2x}$

5. Graph (c), since this is the only graph that has positive slope at c.

6. Graph (b), since this is the only graph that represents a differentiable function at a and b and has negative slope at c.

7. Graph (d), since this is the only graph representing a function that is differentiable at b but not at a.

8. Graph (a), since this is the only graph that represents a function that is not differentiable at a or b.

9. As $x \to 3^-$, $\sqrt{9-x^2} \to 0^+$. Therefore, $\lim\limits_{x \to 3^-} f(x) = \infty$.

10. As $x \to 3^+$, $\sqrt{9-x^2} \to 0^+$. Therefore, $\lim\limits_{x \to -3^+} f(x) = \infty$.

11. **(a)** $\dfrac{d}{dx}(x^3 - 2x) = 3x^2 - 2$
    $f'(1) = 3(1)^2 - 2 = 1$

    (b) $\dfrac{d}{dx}(x+2) = 1$
    $f'(3) = 1$

(c) Left-hand derivative:

$$\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0^-} \frac{[(2+h)^3-2(2+h)]-4}{h}$$
$$= \lim_{h\to 0^-} \frac{h^3+6h^2+10h}{h}$$
$$= \lim_{h\to 0^-} (h^2+6h+10)$$
$$=10$$

Right-hand derivative:

$$\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0^+} \frac{[(2+h)+2]-4}{h}$$
$$= \lim_{h\to 0^+} \frac{h}{h}$$
$$= \lim_{h\to 0^+} 1$$
$$=1$$

Since the left-and right-hand derivatives are not equal, $f'(2)$ is undefined.

12. **(a)** The domain is $x \neq 2$. (See the solution for 11.(c)).

(b) $f'(x) = \begin{cases} 3x^2-2, & x<2 \\ 1, & x>2 \end{cases}$

Section 5.1 Exercises

1. Minima at (–2, 0) and (2, 0), maximum at (0, 2)

2. Local minimum at (–1, 0), local maximum at (1, 0)

3. Maximum at (0, 5); note that there is no minimum since the endpoint (2, 0) is excluded from the graph.

4. Local maximum at (–3, 0), local minimum at (2, 0), maximum at (1, 2), minimum at (0, –1)

5. Maximum at $x = b$, minimum at $x = c_2$;
The Extreme Value Theorem applies because f is continuous on $[a, b]$, so both the maximum and minimum exist.

6. Maximum at $x = c$, minimum at $x = b$;
The Extreme Value Theorem applies because f is continuous on $[a, b]$, so both the maximum and minimum exist.

7. Maximum at $x = c$, no minimum; the Extreme Value Theorem does not apply, because the function is not defined on a closed interval.

8. No maximum, no minimum; the Extreme Value Theorem does not apply, because the function is not continuous or defined on a closed interval.

9. Maximum at $x = c$, minimum at $x = a$; the Extreme Value Theorem does not apply, because the function is not continuous.

10. Maximum at $x = a$, minimum at $x = c$; the Extreme Value Theorem does not apply since the function is not continuous.

11. The first derivative $f'(x) = -\frac{1}{x^2}+\frac{1}{x}$ has a zero at $x=1$.
Critical point value: $f(1) = 1+\ln 1 = 1$
Endpoint values: $f(0.5) = 2+\ln 0.5 \approx 1.307$

$$f(4) = \frac{1}{4}+\ln 4 \approx 1.636$$

Maximum value is $\frac{1}{4}+\ln 4$ at $x = 4$;
minimum value is 1 at $x = 1$;
local maximum at $\left(\frac{1}{2}, 2-\ln 2\right)$
Since f' is zero at the only critical point, there are no critical points that are not stationary points.

12. The first derivative $g'(x) = -e^{-x}$ has no zeros, so we need only consider the endpoints.
$g(-1) = e^{-(-1)} = e$
$g(1) = e^{-1} = \frac{1}{e}$
Maximum value is e at $x = -1$;
minimum value is $\frac{1}{e}$ at $x = 1$.
Since there are no critical points, there are no critical points that are not stationary points.

13. The first derivative $h'(x) = \frac{1}{x+1}$ has no zeros, so we need only consider the endpoints.
$h(0) = \ln 1 = 0 \qquad h(3) = \ln 4$
Maximum value is ln 4 at $x = 3$; minimum value is 0 at $x = 0$.
Since there are no critical points, there are no critical points that are not stationary points.

14. The first derivative $k'(x) = -2xe^{-x^2}$ has a zero at $x = 0$. Since the domain has no endpoints, any extreme value must occur at $x = 0$.
Since $k(0) = e^{-0^2} = 1$ and $\lim_{x\to\pm\infty} k(x) = 0$, the maximum value is 1 at $x = 0$.
Since k' is zero at the only critical point, there are no critical points that are not stationary points.

15. The first derivative $f'(x) = \cos\left(x + \frac{\pi}{4}\right)$, has zeros at $x = \frac{\pi}{4}$, $x = \frac{5\pi}{4}$.

Critical point values: $x = \frac{\pi}{4}$ $\quad f(x) = 1$

$x = \frac{5\pi}{4}$ $\quad f(x) = -1$

Endpoint values: $x = 0$ $\quad f(x) = \frac{1}{\sqrt{2}}$

$x = \frac{7\pi}{4}$ $\quad f(x) = 0$

Maximum value is 1 at $x = \frac{\pi}{4}$;
minimum value is -1 at $x = \frac{5\pi}{4}$;
local minimum at $\left(0, \frac{1}{\sqrt{2}}\right)$;
local maximum at $\left(\frac{7\pi}{4}, 0\right)$
Since f' is zero at both the critical points, there are no critical points that are not stationary points.

16. The first derivative $g'(x) = \sec x \tan x$ has zeros at $x = 0$ and $x = \pi$ and is undefined at $x = \frac{\pi}{2}$.
Since $g(x) = \sec x$ is also undefined at $x = \frac{\pi}{2}$, the critical points occur only at $x = 0$ and $x = \pi$.

Critical point values: $x = 0$ $\quad g(x) = 1$

$x = \pi$ $\quad g(x) = -1$

Since the range of $g(x)$ is $(-\infty, -1] \cup [1, \infty)$, these values must be a local minimum and local maximum, respectively. Local minimum at $(0, 1)$; local maximum at $(\pi, -1)$
Since g' is zero at both the critical points, there are no critical points that are not stationary points.

17. The first derivative $f'(x) = \frac{2}{5}x^{-3/5}$ is never zero but is undefined at $x = 0$.

Critical point value: $x = 0$ $\quad f(x) = 0$

Endpoint value: $x = -3$ $\quad f(x) = (-3)^{2/5} = 3^{2/5} \approx 1.552$

Since $f(x) > 0$ for $x \neq 0$, the critical point at $x = 0$ is a local minimum, and since $f(x) \le (-3)^{2/5}$ for $-3 \le x < 1$, the endpoint value at $x = -3$ is a global maximum.
Maximum value is $3^{2/5}$ at $x = -3$;
minimum value is 0 at $x = 0$.
Since f' is undefined at $x = 0$, the critical point $(0, 0)$ is not a stationary point.

18. The first derivative $f'(x) = \frac{3}{5}x^{-2/5}$ is never zero but is undefined at $x = 0$.

Critical point value: $x = 0$ $\quad f(x) = 0$

Endpoint value: $x = 3$ $\quad f(x) = 3^{3/5} \approx 1.933$

Since $f(x) < 0$ for $x < 0$ and $f(x) > 0$ for $x > 0$, the critical point is not a local minimum or maximum. The maximum value is $3^{3/5}$ at $x = 3$. Since f' is undefined at $x = 0$, the critical point $(0, 0)$ is not a stationary point.

19. The domain is $(-\infty, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.
$y' = 4x - 8$
The only critical point is $x = 2$. The value $y = 2(2)^2 - 8(2) + 9 = 1$ is the only candidate for an extreme value. As x moves away from 2 on either side, the values of y increase, and the graph rises. We have a minimum value of 1 at $x = 2$.

20. The domain is $(-\infty, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.
$y' = 3x^2 - 2$
The critical points are $\pm\sqrt{\frac{2}{3}}$. The values

$y = \left(\sqrt{\frac{2}{3}}\right)^3 - 2\sqrt{\frac{2}{3}} + 4 = 4 - \frac{4\sqrt{6}}{9}$ and

$y = \left(-\sqrt{\frac{2}{3}}\right)^3 - 2\left(-\sqrt{\frac{2}{3}}\right) + 4 = 4 + \frac{4\sqrt{6}}{9}$ are the only candidates for extreme values. As x moves away from $-\sqrt{\frac{2}{3}}$ on either side, the values of y decrease. The function has a local maximum at $\left(-\sqrt{\frac{2}{3}},\ 4 + \frac{4\sqrt{6}}{9}\right) \approx (-0.816, 5.089)$. As x moves away from $\sqrt{\frac{2}{3}}$ on either side, the values of y increase. The function has a local minimum at $\left(\sqrt{\frac{2}{3}},\ 4 - \frac{4\sqrt{6}}{9}\right) \approx (0.816, 2.911)$.

21. The domain is $(-\infty, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.

$y' = 3x^2 + 2x - 8 = (3x - 4)(x + 2)$

The critical points are $\frac{4}{3}$ and -2. The values

$y = \left(\frac{4}{3}\right)^3 + \left(\frac{4}{3}\right)^2 - 8\left(\frac{4}{3}\right) + 5 = -\frac{41}{27}$ and

$y = (-2)^3 + (-2)^2 - 8(-2) + 5 = 17$ are the only candidates for extreme values. As x moves away from $\frac{4}{3}$ on either side, the values of y increase. The function has a local minimum at $\left(\frac{4}{3}, -\frac{41}{27}\right)$. As x moves away from -2 on either side, the values of y decrease. The function has a local maximum at $(-2, 17)$.

22. The domain is $(-\infty, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.

$y' = 3x^2 - 6x + 3 = 3(x - 1)^2$

The only critical point is $x = 1$. The value $y = (1)^3 - 3(1)^2 + 3(1) - 2 = -1$ is the only candidate for an extreme value. As x moves away from 1 on the left, the values of y decrease. As x moves away from 1 on the right, the values of y increase. Neither a local maximum nor a local minimum occurs at $x = 1$. There are no local maxima or minima.

23. The domain is $(-\infty, -1] \cup [1, \infty)$.

$y' = \frac{1}{2}(x^2 - 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 - 1}}$

The derivative is zero only when $x = 0$, which is not in the domain. The derivative is undefined at $x = \pm 1$, which are also the endpoints. As x moves away from ± 1 within the domain, the values of y increase. The function has a minimum value of 0 at $x = -1$ and at $x = 1$.

24. The domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.

$y' = -1(x^2 - 1)^{-2}(2x) = -\frac{2x}{(x^2 - 1)^2}$

The derivative is zero only when $x = 0$. The derivative is undefined at $x = \pm 1$, which are not in the domain. The only critical point is $x = 0$. As x moves away from 0 on either side, the values of y decrease. The function has a local maximum value at $(0, -1)$.

25. The domain is $(-1, 1)$. The domain has no endpoints, so all the extreme values must occur at critical points.

$y' = -\frac{1}{2}(1 - x^2)^{-3/2}(-2x) = \frac{x}{(1 - x^2)^{3/2}}$

The derivative is zero only when $x = 0$. The derivative is undefined at $x = \pm 1$, which are not in the domain. The only critical point is $x = 0$. As x moves away from 0 on either side, the values of y increase. The function has a minimum value of 1 at $x = 0$.

26. The domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.

$y' = -\frac{1}{3}(1 - x^2)^{-4/3}(-2x) = \frac{2x}{3(1 - x^2)^{4/3}}$

The derivative is zero only when $x = 0$. The derivative is undefined at $x = \pm 1$, which are not in the domain. The only critical point is $x = 0$. As x moves away from 0 on either side, the values of y increase. The function has a local minimum value at $(0, 1)$.

27. The domain is $[-1, 3]$.

$$y' = \frac{1}{2}(3 + 2x - x^2)^{-1/2}(2 - 2x) = \frac{1 - x}{\sqrt{3 + 2x - x^2}}$$

The derivative is zero when $x = 1$. The

derivative is undefined at $x = -1$ and at $x = 3$, which are also the endpoints. As x moves away from 1 on either side, the values of y decrease. The function has a maximum value of 2 at $x = 1$. As x moves away from -1 or 3 within the domain, the values of y increase. The function has a minimum value of 0 at $x = -1$ and at $x = 3$.

28. The domain is $(-\infty, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.

$$y' = 6x^3 + 12x^2 - 18x = 6x(x+3)(x-1)$$

The critical points are 0, -3, and 1. As x moves away from 0 on either side, the values of y decrease. The function has a local maximum at (0, 10). As x moves away from -3 on either side, the values of y increase. The function has a minimum value of $-\frac{115}{2}$ at $x = -3$. As x moves away from 1 on either side, the values of y increase. The function has a local minimum at $\left(1, \frac{13}{2}\right)$.

29. The domain is $(-\infty, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.

$$y' = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

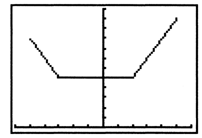
The critical points are -1 and 1. As x moves away from -1 on either side, the values of y increase. The function has a minimum value of $-\frac{1}{2}$ at $x = -1$. As x moves away from 1 on either side, the values of y decrease. The function has a maximum value of $\frac{1}{2}$ at $x = 1$.

30. The domain is $(-\infty, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.

$$y' = \frac{(x^2+2x+2)(1) - (x+1)(2x+2)}{(x^2+2x+2)^2} = \frac{-x(x+2)}{(x^2+2x+2)^2}$$

The critical points are -2 and 0. As x moves away from -2 on either side, the values of y increase. The function has a minimum value of $-\frac{1}{2}$ at $x = -2$. As x moves away from 0 on either side, the values of y decrease. The function has a maximum value of $\frac{1}{2}$ at $x = 0$.

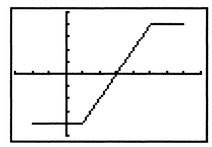
31.



[−6, 6] by [0, 12]

Maximum value is 11 at $x = 5$;
minimum value is 5 on the interval $[-3, 2]$;
local maximum at $(-5, 9)$

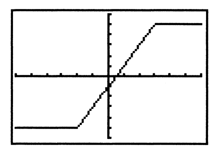
32.



[−3, 8] by [−5, 5]

Maximum value is 4 on the interval [5, 7];
minimum value is -4 on the interval $[-2, 1]$.

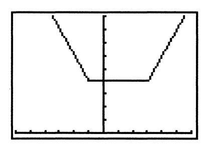
33.



[−6, 6] by [−6, 6]

Maximum value is 5 on the interval $[3, \infty)$;
minimum value is -5 on the interval $(-\infty, -2]$.

34.



[−6, 6] by [0, 9]

Minimum value is 4 on the interval $[-1, 3]$

35. $y' = x^{2/3}(1) + \frac{2}{3}x^{-1/3}(x+2) = \frac{5x+4}{3\sqrt[3]{x}}$

crit. pt.	derivative	extremum	value
$x = -\frac{4}{5}$	0	local max	$\frac{12}{25}10^{1/3} \approx 1.034$
$x = 0$	undefined	local min	0

Since y' is undefined at $x = 0$, the critical point (0, 0) is not a stationary point.

36. $y' = x^{2/3}(2x) + \frac{2}{3}x^{-1/3}(x^2 - 4) = \frac{8x^2 - 8}{3\sqrt[3]{x}}$

crit. pt.	derivative	extremum	value
$x = -1$	0	minimum	–3
$x = 0$	undefined	local max	0
$x = 1$	0	minimum	–3

Since y' is undefined at $x = 0$, the critical point (0, 0) is not a stationary point.

37.
$$y' = x \cdot \frac{1}{2\sqrt{4-x^2}}(-2x) + (1)\sqrt{4-x^2}$$
$$= \frac{-x^2 + (4 - x^2)}{\sqrt{4-x^2}}$$
$$= \frac{4 - 2x^2}{\sqrt{4-x^2}}$$

crit. pt.	derivative	extremum	value
$x = -2$	undefined	local max	0
$x = -\sqrt{2}$	0	minimum	–2
$x = \sqrt{2}$	0	maximum	2
$x = 2$	undefined	local min	0

Since y' is undefined at $x = -2$ and $x = 2$, the critical points (–2, 0) and (2, 0) are not stationary points.

38.
$$y = x^2 \cdot \frac{1}{2\sqrt{3-x}}(-1) + 2x\sqrt{3-x}$$
$$= \frac{-x^2 + 4x(3-x)}{2\sqrt{3-x}}$$
$$= \frac{-5x^2 + 12x}{2\sqrt{3-x}}$$

crit. pt.	derivative	extremum	value
$x = 0$	0	minimum	0
$x = \frac{12}{5}$	0	local max	$\frac{144}{125}15^{1/2}$ ≈ 4.462
$x = 3$	undefined	minimum	0

Since y' is undefined at $x = 3$, the critical point (3, 0) is not a stationary point.

39. $y' = \begin{cases} -2, & x < 1 \\ 1, & x > 1 \end{cases}$

crit. pt.	derivative	extremum	value
$x = 1$	undefined	minimum	2

Since y' is undefined at $x = 1$, the critical point (1, 2) is not a stationary point.

40. $y' = \begin{cases} -1, & x < 0 \\ 2 - 2x, & x > 0 \end{cases}$

crit. pt.	derivative	extremum	value
$x = 0$	undefined	local min	3
$x = 1$	0	local max	4

Since y' is undefined at $x = 0$, the critical point (0, 3) is not a stationary point.

41. $y' = \begin{cases} -2x - 2, & x < 1 \\ -2x + 6, & x > 1 \end{cases}$

crit. pt.	derivative	extremum	value
$x = -1$	0	maximum	5
$x = 1$	undefined	local min	1
$x = 3$	0	maximum	5

Since y' is undefined at $x = 1$, the critical point (1, 1) is not a stationary point.

42. We begin by determining whether $f'(x)$ is defined at $x = 1$, where

$$f(x) = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$$

Left-hand derivative:

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h \to 0^-} \frac{-\frac{1}{4}(1+h)^2 - \frac{1}{2}(1+h) + \frac{15}{4} - 3}{h}$$
$$= \lim_{h \to 0^-} \frac{-h^2 - 4h}{4h}$$
$$= \lim_{h \to 0^-} \frac{1}{4}(-h - 4)$$
$$= -1$$

Right-hand derivative:

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0^+}\frac{(1+h)^3-6(1+h)^2+8(1+h)-3}{h}$$
$$=\lim_{h\to 0^+}\frac{h^3-3h^2-h}{h}$$
$$=\lim_{h\to 0^+}(h^2-3h-1)$$
$$=-1$$

Thus $f'(x)=\begin{cases}-\frac{1}{2}x-\frac{1}{2}, & x\le 1\\ 3x^2-12x+8, & x>1\end{cases}$

Note that $-\frac{1}{2}x-\frac{1}{2}=0$ when $x=-1$, and

$3x^2-12x+8=0$ when $x=2\pm\frac{2\sqrt{3}}{3}$.

But $2-\frac{2\sqrt{3}}{3}\approx 0.845<1$, so the only critical points occur at $x=-1$ and

$x=2+\frac{2\sqrt{3}}{3}\approx 3.155$.

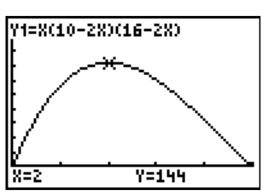
crit. pt.	derivative	extremum	value
$x=-1$	0	local max	4
$x\approx 3.155$	0	local min	≈ -3.079

Since y' is zero at both the critical points, there are no critical points that are not stationary points.

43. **(a)** $V(x)=160x-52x^2+4x^3$
$$V'(x)=160-104x+12x^2$$
$$=4(x-2)(3x-20)$$
The only critical point in the interval (0, 5) is at $x=2$. The maximum value of $V(x)$ is 144 at $x=2$.

(b) The largest possible volume of the box is 144 cubic units, and it occurs when $x=2$.

(c)



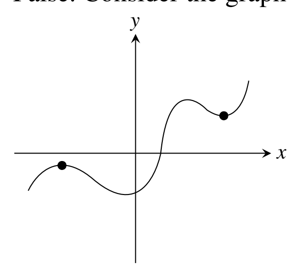
[0, 5] by [–20, 200]

44. **(a)** $P'(x)=2-200x^{-2}$
The only critical point in the interval $(0, \infty)$ is at $x=10$. The minimum value of $P(x)$ is 40 at $x=10$.

(b) The smallest possible perimeter of the rectangle is 40 units and it occurs at $x=10$, which makes the rectangle a 10 by 10 square.

45. False; for example, the maximum could occur at a corner, where $f'(c)$ would not exist.

46. False. Consider the graph below.



47. E; $\frac{d}{dx}(4x-x^2+6)=4-2x$
$$4-2x=0$$
$$x=2$$
$$f(2)=4(2)-(2)^2+6=10$$

48. E; see Theorem 2.

49. B’ $\frac{d}{dx}(x^3-6x+5)=3x^2-6$
$$3x^2-6=0$$
$$x=\pm\sqrt{2}$$

50. B

51. **(a)** No, since $f'(x)=\frac{2}{3}(x-2)^{-1/3}$, which is undefined at $x=2$.

(b) The derivative is defined and nonzero for all $x\ne 2$. Also, $f(2)=0$ and $f(x)>0$ for all $x\ne 2$.

(c) No, $f(x)$ need not have a global maximum because its domain is all real numbers. Any restriction of f to a closed interval of the form $[a, b]$ would have both a maximum value and a minimum value on the interval.

(d) The answers are the same as (a) and (b) with 2 replaced by *a*.

52. Note that

$$f(x) = \begin{cases} -x^3 + 9x, & x \le -3 \text{ or } 0 \le x < 3 \\ x^3 - 9x, & -3 < x < 0 \text{ or } x \ge 3. \end{cases}$$

Therefore,

$$f'(x) = \begin{cases} -3x^2 + 9, & x < -3 \text{ or } 0 < x < 3 \\ 3x^2 - 9, & -3 < x < 0 \text{ or } x > 3. \end{cases}$$

(a) No, since the left- and right-hand derivatives at $x = 0$ are -9 and 9, respectively.

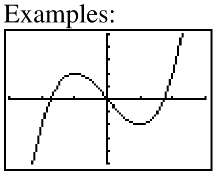
(b) No, since the left- and right-hand derivatives at $x = 3$ are -18 and 18, respectively.

(c) No, since the left- and right-hand derivatives at $x = -3$ are -18 and 18, respectively.

(d) The critical points occur when $f'(x) = 0$ $\left(\text{at } x = \pm\sqrt{3}\right)$ and when $f'(x)$ is undefined (at $x = 0$ or $x = \pm 3$). The minimum value is 0 at $x = -3$, at $x = 0$, and at $x = 3$; local maxima occur at $\left(-\sqrt{3}, 6\sqrt{3}\right)$ and $\left(\sqrt{3}, 6\sqrt{3}\right)$.

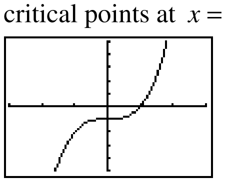
53. (a) $f'(x) = 3ax^2 + 2bx + c$ is a quadratic, so it can have 0, 1, or 2 zeros, which would be the critical points of *f*.

Examples:



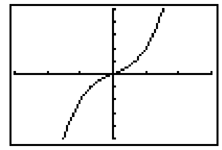
[–3, 3] by [–5, 5]

The function $f(x) = x^3 - 3x$ has two critical points at $x = -1$ and $x = 1$.



[–3, 3] by [–5, 5]

The function $f(x) = x^3 - 1$ has one critical point at $x = 0$.



[–3, 3] by [–5, 5]

The function $f(x) = x^3 + x$ has no critical points.

(b) The function can have either two local extreme values or no extreme values. (If there is only one critical point, the cubic function has no extreme values.)

54. (a) By the definition of local maximum value, there is an open interval containing *c* where $f(x) \le f(c)$, so $f(x) - f(c) \le 0$.

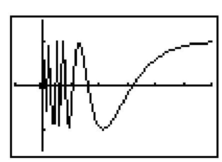
(b) Because $x \to c^+$, we have $(x - c) > 0$, and the sign of the quotient must be negative (or zero). This means the limit is nonpositive.

(c) Because $x \to c^-$, we have $(x - c) < 0$, and the sign of the quotient must be positive (or zero). This means the limit is nonnegative.

(d) Assuming that $f'(c)$ exists, the one-sided limits in (b) and (c) above must exist and be equal. Since one is nonpositive and one is nonnegative, the only possible common value is 0.

(e) There will be an open interval containing *c* where $f(x) - f(c) \ge 0$. The difference quotient for the left-hand derivative will have to be negative (or zero), and the difference quotient for the right-hand derivative will have to be positive (or zero). Taking the limit, the left-hand derivative will be nonpositive, and the right-hand derivative will be nonnegative. Therefore, the only possible value for $f'(c)$ is 0.

55. (a)



[–0.1, 0.6] by [–1.5, 1.5]

$f(0) = 0$ is not a local extreme value because in any open interval containing $x = 0$, there are infinitely many points where $f(x) = 1$ and where $f(x) = -1$.

(b) One possible answer, on the interval [0, 1]:

$$f(x)=\begin{cases}(1-x)\cos\dfrac{1}{1-x}, & 0\le x<1\\ 0, & x=1\end{cases}$$

This function has no local extreme value at $x = 1$. Note that it is continuous on [0, 1].

Section 5.2 Mean Value Theorem (pp. 200–208)

Quick Review 5.2

1. $2x^2-6<0$
$2x^2<6$
$x^2<3$
$-\sqrt{3}<x<\sqrt{3}$
Interval: $(-\sqrt{3}, \sqrt{3})$

2. $3x^2-6>0$
$3x^2>6$
$x^2>2$
$x<-\sqrt{2}$ or $x>\sqrt{2}$
Intervals: $(-\infty, -\sqrt{2})\cup(\sqrt{2}, \infty)$

3. Domain: $8-2x^2\ge 0$
$8\ge 2x^2$
$4\ge x^2$
$-2\le x\le 2$
The domain is [–2, 2].

4. f is continuous for all x in the domain, or, in the interval [–2, 2].

5. f is differentiable for all x in the interior of its domain, or, in the interval (–2, 2).

6. We require $x^2-1\ne 0$, so the domain is $x\ne\pm 1$.

7. f is continuous for all x in the domain, or, for all $x\ne\pm 1$.

8. f is differentiable for all x in the domain, or, for all $x\ne\pm 1$.

9. $7=-2(-2)+C$
$7=4+C$
$C=3$

10. $-1=(1)^2+2(1)+C$
$-1=3+C$
$C=-4$

Section 5.2 Exercises

1. (a) Yes.

(b) $f'(x)=\dfrac{d}{dx}(x^2+2x-1)=2x+2$

$$2c+2=\frac{2-(-1)}{1-0}=3$$
$$c=\frac{1}{2}.$$

2. (a) Yes.

(b) $f'(x)=\dfrac{d}{dx}x^{2/3}=\dfrac{2}{3}x^{-1/3}$

$$\frac{2}{3}c^{-1/3}=\frac{1-0}{1-0}=1$$
$$c=\frac{8}{27}.$$

3. (a) No. There is a vertical tangent at $x = 0$.

4. (a) No. There is a corner at $x = 1$.

5. (a) Yes.

(b) $f'(x)=\dfrac{d}{dx}\sin^{-1}x=\dfrac{1}{\sqrt{1-x^2}}$

$$\frac{1}{\sqrt{1-c^2}}=\frac{(\pi/2)-(-\pi/2)}{1-(-1)}=\frac{\pi}{2}$$
$$\sqrt{1-c^2}=\frac{2}{\pi}$$
$$c=\sqrt{1-4/\pi^2}\approx 0.771.$$

6. (a) Yes.

(b) $f'(x)=\dfrac{d}{dx}\ln(x-1)=\dfrac{1}{x-1}$

$$\frac{1}{c-1}=\frac{\ln 3-\ln 1}{4-2}$$
$$c=\frac{4-2}{\ln 3-\ln 1}+1\approx 2.820$$

7. (a) No; the function is discontinuous at $x=\dfrac{\pi}{2}$.

8. **(a)** No; the function is discontinuous at $x = 1$.

9. **(a)** The secant line passes through $(0.5, f(0.5)) = (0.5, 2.5)$ and $(2, f(2)) = (2, 2.5)$, so its equation is $y = 2.5$.

(b) The slope of the secant line is 0, so we need to find c such that $f'(c) = 0$.

$$1 - c^{-2} = 0$$
$$c^{-2} = 1$$
$$c = 1$$
$$f(c) = f(1) = 2$$

The tangent line has slope 0 and passes through (1, 2), so its equation is $y = 2$.

10. **(a)** The secant line passes through $(1, f(1)) = (1, 0)$ and $(3, f(3)) = (3, \sqrt{2})$, so its slope is $\dfrac{\sqrt{2}-0}{3-1} = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$.

The equation is $y = \dfrac{1}{\sqrt{2}}(x-1)+0$ or $y = \dfrac{1}{\sqrt{2}}x - \dfrac{1}{\sqrt{2}}$, or $y \approx 0.707x - 0.707$.

(b) We need to find c such that $f'(c) = \dfrac{1}{\sqrt{2}}$.

$$\frac{1}{2\sqrt{c-1}} = \frac{1}{\sqrt{2}}$$
$$2\sqrt{c-1} = \sqrt{2}$$
$$c - 1 = \frac{1}{2}$$
$$c = \frac{3}{2}$$
$$f(c) = f\left(\frac{3}{2}\right) = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

The tangent line has slope $\dfrac{1}{\sqrt{2}}$ and passes through $\left(\dfrac{3}{2}, \dfrac{1}{\sqrt{2}}\right)$. Its equation is

$y = \dfrac{1}{\sqrt{2}}\left(x - \dfrac{3}{2}\right) + \dfrac{1}{\sqrt{2}}$ or

$y = \dfrac{1}{\sqrt{2}}x - \dfrac{1}{2\sqrt{2}}$, or $y \approx 0.707x - 0.354$.

11. Because the trucker's average speed was 79.5 mph, and by the Mean Value Theorem, the trucker must have been going that speed at least once during the trip.

12. Let $f(t)$ denote the temperature indicated after t seconds. We assume that $f'(t)$ is defined and continuous for $0 \le t \le 20$. The average rate of change is $10.6°$ F/sec. Therefore, by the Mean Value Theorem, $f'(c) = 10.6°$ F/sec for some value of c in [0, 20]. Since the temperature was constant before $t = 0$, we also know that $f'(0) = 0°$ F/min. But f' is continuous, so by the Intermediate Value Theorem, the rate of change $f'(t)$ must have been 10.6°F/sec at some moment during the interval.

13. Because its average speed was approximately 7.667 knots, and by the Mean Value Theorem, it must been going that speed at least once during the trip.

14. The runner's average speed for the marathon was approximately 11.909 mph. Therefore, by the Mean Value Theorem, the runner must have been going that speed at least once during the marathon. Since the initial speed and final speed are both 0 mph and the runner's speed is continuous, by the Intermediate Value Theorem, the runner's speed must have been 11 mph at least twice.

15. **(a)** $f'(x) = 5 - 2x$

Since $f'(x) > 0$ on $\left(-\infty, \dfrac{5}{2}\right)$, $f'(x) = 0$ at $x = \dfrac{5}{2}$, and $f'(x) < 0$ on $\left(\dfrac{5}{2}, \infty\right)$, we know that $f(x)$ has a local maximum at $x = \dfrac{5}{2}$. Since $f\left(\dfrac{5}{2}\right) = \dfrac{25}{4}$, the local maximum occurs at the point $\left(\dfrac{5}{2}, \dfrac{25}{4}\right)$. (This is also a global maximum.)

(b) Since $f'(x) > 0$ on $\left(-\infty, \dfrac{5}{2}\right)$, $f(x)$ is increasing on $\left(-\infty, \dfrac{5}{2}\right]$.

(c) Since $f'(x) < 0$ on $\left(\frac{5}{2}, \infty\right)$, $f(x)$ is decreasing on $\left[\frac{5}{2}, \infty\right)$.

16. (a) $g'(x) = 2x - 1$

Since $g'(x) < 0$ on $\left(-\infty, \frac{1}{2}\right)$, $g'(x) = 0$ at $x = \frac{1}{2}$, and $g'(x) > 0$ on $\left(\frac{1}{2}, \infty\right)$, we know that $g(x)$ has a local minimum at $x = \frac{1}{2}$.

Since $g\left(\frac{1}{2}\right) = -\frac{49}{4}$, the local minimum occurs at the point $\left(\frac{1}{2}, -\frac{49}{4}\right)$. (This is also a global minimum.)

(b) Since $g'(x) > 0$ on $\left(\frac{1}{2}, \infty\right)$, $g(x)$ is increasing on $\left[\frac{1}{2}, \infty\right)$.

(c) Since $g'(x) < 0$ on $\left(-\infty, \frac{1}{2}\right)$, $g(x)$is decreasing on $\left(-\infty, \frac{1}{2}\right]$.

17. (a) $h'(x) = -\frac{2}{x^2}$

Since $h'(x)$ is never zero and is undefined only where $h(x)$ is undefined, there are no critical points. Also, the domain $(-\infty, 0) \cup (0, \infty)$ has no endpoints. Therefore, $h(x)$ has no local extrema.

(b) Since $h'(x)$ is never positive, $h(x)$ is not increasing on any interval.

(c) Since $h'(x) < 0$ on $(-\infty, 0) \cup (0, \infty)$, $h(x)$ is decreasing on $(-\infty, 0)$ and on $(0, \infty)$.

18. (a) $k'(x) = -\frac{2}{x^3}$

Since $k'(x)$ is never zero and is undefined only where $k(x)$ is undefined, there are no critical points. Also, the domain $(-\infty, 0) \cup (0, \infty)$ has no endpoints. Therefore, $k(x)$ has no local extrema.

(b) Since $k'(x) > 0$ on $(-\infty, 0)$, $k(x)$ is increasing on $(-\infty, 0)$.

(c) Since $k'(x) < 0$ on $(0, \infty)$, $k(x)$ is decreasing on $(0, \infty)$.

19. (a) $f'(x) = 2e^{2x}$

Since $f'(x)$ is never zero or undefined, and the domain of $f(x)$ has no endpoints, $f(x)$ has no extrema.

(b) Since $f'(x)$ is always positive, $f(x)$ is increasing on $(-\infty, \infty)$.

(c) Since $f'(x)$ is never negative, $f(x)$ is not decreasing on any interval.

20. (a) $f'(x) = -0.5e^{-0.5x}$

Since $f'(x)$ is never zero or undefined, and the domain of $f(x)$ has no endpoints, $f(x)$ has no extrema.

(b) Since $f'(x)$ is never positive, $f(x)$ is not increasing on any interval.

(c) Since $f'(x)$ is always negative, $f(x)$ is decreasing on $(-\infty, \infty)$.

21. (a) $y' = -\frac{1}{2\sqrt{x+2}}$

In the domain $[-2, \infty)$, y' is never zero and is undefined only at the endpoint $x = -2$. The function y has a local maximum at $(-2, 4)$. (This is also a global maximum.)

(b) Since y' is never positive, y is not increasing on any interval.

(c) Since y' is negative on $(-2, \infty)$, y is decreasing on $[-2, \infty)$.

22. (a) $y' = 4x^3 - 20x = 4x\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)$

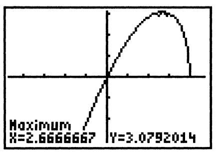
The function has critical points at $x = -\sqrt{5}$, $x = 0$, and $x = \sqrt{5}$. Since $y' < 0$ on $\left(-\infty, -\sqrt{5}\right)$ and $\left(0, \sqrt{5}\right)$ and $y' > 0$ on $\left(-\sqrt{5}, 0\right)$ and $\left(\sqrt{5}, \infty\right)$, the points at $x = \pm\sqrt{5}$ are local minima and the point at $x = 0$ is a local maximum.

Thus, the function has a local maximum at (0, 9) and local minima at $\left(-\sqrt{5}, -16\right)$ and $\left(\sqrt{5}, -16\right)$. (These are also global minima.)

(b) Since $y' > 0$ on $\left(-\sqrt{5}, 0\right)$ and $\left(\sqrt{5}, \infty\right)$, y is increasing on $\left[-\sqrt{5}, 0\right]$ and $\left[\sqrt{5}, \infty\right)$.

(c) Since $y' > 0$ on $\left(-\infty, -\sqrt{5}\right)$ and $\left(0, \sqrt{5}\right)$, y is decreasing on $\left(-\infty, -\sqrt{5}\right)$ and $\left[0, \sqrt{5}\right]$.

23.



[−4.7, 4.7] by [−3.1, 3.1]

(a) $f'(x) = x \cdot \dfrac{1}{2\sqrt{4-x}}(-1) + \sqrt{4-x}$

$= \dfrac{-3x+8}{2\sqrt{4-x}}$

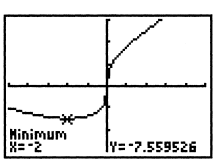
The local extrema occur at the critical point $x = \dfrac{8}{3}$ and at the endpoint $x = 4$.

There is a local (and absolute) maximum at $\left(\dfrac{8}{3}, \dfrac{16}{3\sqrt{3}}\right)$ or approximately (2.67, 3.08), and a local minimum at (4, 0).

(b) Since $f'(x) > 0$ on $\left(-\infty, \dfrac{8}{3}\right)$, $f(x)$ is increasing on $\left(-\infty, \dfrac{8}{3}\right]$.

(c) Since $f'(x) < 0$ on $\left(\dfrac{8}{3}, 4\right)$, $f(x)$ is decreasing on $\left[\dfrac{8}{3}, 4\right]$.

24.



[−5, 5] by [−15, 15]

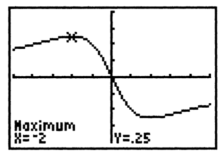
(a) $g'(x) = x^{1/3}(1) + \dfrac{1}{3}x^{-2/3}(x+8) = \dfrac{4x+8}{3x^{2/3}}$

The local extrema can occur at the critical points $x = -2$ and $x = 0$, but the graph shows that no extrema occurs at $x = 0$.
There is a local (and absolute) minimum at $(-2, -6\sqrt[3]{2})$ or approximately (–2, –7.56).

(b) Since $g'(x) > 0$ on the intervals $(-2, 0)$ and $(0, \infty)$, and $g(x)$ is continuous at $x = 0$, $g(x)$ is increasing on $[-2, \infty)$.

(c) Since $g'(x) < 0$ on the interval $(-\infty, -2)$, $g(x)$ is decreasing on $(-\infty, -2]$.

25.



[−5, 5] by [−0.4, 0.4]

(a) $h'(x) = \dfrac{(x^2+4)(-1) - (-x)(2x)}{(x^2+4)^2}$

$= \dfrac{x^2-4}{(x^2+4)^2}$

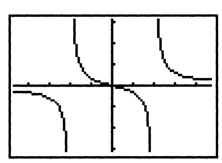
$= \dfrac{(x+2)(x-2)}{(x^2+4)^2}$

The local extrema occur at the critical points, $x = \pm 2$.
There is a local (and absolute) maximum at $\left(-2, \dfrac{1}{4}\right)$ and a local (and absolute) minimum at $\left(2, -\dfrac{1}{4}\right)$.

(b) Since $h'(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $h(x)$ is increasing on $(-\infty, -2]$ and $[2, \infty)$.

(c) Since $h'(x) < 0$ on $(-2, 2)$, $h(x)$ is decreasing on $[-2, 2]$.

26.



[−4.7, 4.7] by [−3.1, 3.1]

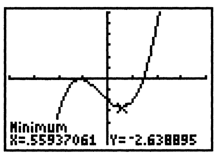
(a) $k'(x) = \dfrac{(x^2-4)(1)-x(2x)}{(x^2-4)^2} = -\dfrac{x^2+4}{(x^2-4)^2}$

Since $k'(x)$ is never zero and is undefined only where $k(x)$ is undefined, there are no critical points. Since there are no critical points and the domain includes no endpoints, $k(x)$ has no local extrema.

(b) Since $k'(x)$ is never positive, $k(x)$ is not increasing on any interval.

(c) Since $k'(x)$ is negative wherever it is defined, $k(x)$ is decreasing on each interval of its domain; on $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

27.



[−4, 4] by [−6, 6]

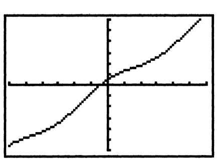
(a) $f'(x) = 3x^2 - 2 + 2\sin x$

Note that $3x^2 - 2 > 2$ for $|x| \ge 1.2$ and $|2 \sin x| \le 2$ for all x, so $f'(x) > 0$ for $|x| \ge 1.2$. Therefore, all critical points occur in the interval $(-1.2, 1.2)$, as suggested by the graph. Using grapher techniques, there is a local maximum at approximately $(-1.126, -0.036)$, and a local minimum at approximately $(0.559, -2.639)$.

(b) $f(x)$ is increasing on the intervals $(-\infty, -1.126]$ and $[0.559, \infty)$, where the interval endpoints are approximate.

(c) $f(x)$ is decreasing on the interval $[-1.126, 0.559]$, where the interval endpoints are approximate.

28.



[−6, 6] by [−12, 12]

(a) $g'(x) = 2 - \sin x$
Since $1 \le g'(x) \le 3$ for all x, there are no critical points. Since there are no critical points and the domain has no endpoints, there are no local extrema.

(b) Since $g'(x) > 0$ for all x, $g(x)$ is increasing on $(-\infty, \infty)$.

(c) Since $g'(x)$ is never negative, $g(x)$ is not decreasing on any interval.

29. $f(x) = \dfrac{x^2}{2} + C$

30. $f(x) = 2x + C$

31. $f(x) = x^3 - x^2 + x + C$

32. $f(x) = -\cos x + C$

33. $f(x) = e^x + C$

34. $f(x) = \ln\,(x-1) + C$

35. $f(x) = \dfrac{1}{x} + C,\ x > 0$

$$f(2) = 1$$
$$\frac{1}{2} + C = 1$$
$$C = \frac{1}{2}$$
$$f(x) = \frac{1}{x} + \frac{1}{2},\ x > 0$$

36. $f(x) = x^{1/4} + C$

$$f(1) = -2$$
$$1^{1/4} + C = -2$$
$$1 + C = -2$$
$$C = -3$$
$$f(x) = x^{1/4} - 3$$

37. $f(x) = \ln(x+2) + C$

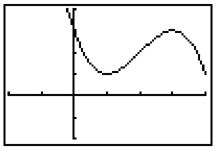
$$f(-1) = 3$$
$$\ln(-1+2) + C = 3$$
$$0 + C = 3$$
$$C = 3$$
$$f(x) = \ln(x+2) + 3$$

38. $f(x) = x^2 + x - \sin x + C$

$$f(0) = 3$$
$$0 + C = 3$$
$$C = 3$$
$$f(x) = x^2 + x - \sin x + 3$$

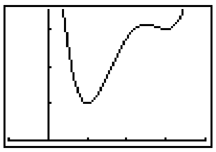
39. Possible answers:

(a)



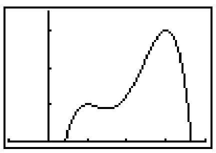
[–2, 4] by [–2, 4]

(b)



[–1, 4] by [0, 3.5]

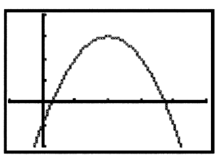
(c)



[–1, 4] by [0, 3.5]

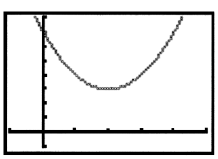
40. Possible answers:

(a)



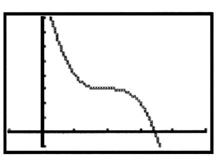
[−1, 5] by [−2, 4]

(b)



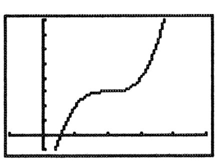
[−1, 5] by [−1, 8]

(c)



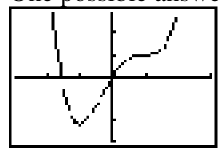
[−1, 5] by [−1, 8]

(d)



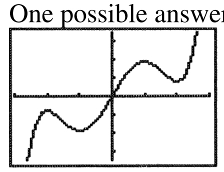
[−1, 5] by [−1, 8]

41. One possible answer:



[–3, 3] by [–15, 15]

42. One possible answer:



[−3, 3] by [−70, 70]

43. **(a)** Since $v'(t) = 1.6$, $v(t) = 1.6t + C$. But $v(0) = 0$, so $C = 0$ and $v(t) = 1.6t$. Therefore, $v(30) = 1.6(30) = 48$. The rock will be going 48 m/sec.

(b) Let $s(t)$ represent position. Since $s'(t) = v(t) = 1.6t$, $s(t) = 0.8t^2 + D$. But $s(0) = 0$, so $D = 0$ and $s(t) = 0.8t^2$. Therefore, $s(30) = 0.8(30)^2 = 720$. The rock travels 720 meters in the 30 seconds it takes to hit bottom, so the bottom of the crevasse is 720 meters below the point of release.

(c) The velocity is now given by $v(t) = 1.6t + C$, where $v(0) = 4$. (Note that the sign of the initial velocity is the same as the sign used for the acceleration, since both act in a downward direction.) Therefore, $v(t) = 1.6t + 4$, and $s(t) = 0.8t^2 + 4t + D$, where $s(0) = 0$ and so $D = 0$. Using $s(t) = 0.8t^2 + 4t$ and the known crevasse depth of 720 meters, we solve $s(t) = 720$ to obtain the positive solution $t \approx 27.604$, and so

$$\begin{aligned} v(t) &= v(27.604) \\ &= 1.6(27.604) + 4 \\ &\approx 48.166. \end{aligned}$$

The rock will hit bottom after about 27.604 seconds, and it will be going about 48.166 m/sec.

44. **(a)** We assume the diving board is located at $s = 0$ and the water at $s = 10$, so that downward velocities are positive. The acceleration due to gravity is 9.8 m/sec^2, so $v'(t) = 9.8$ and $v(t) = 9.8t + C$. Since $v(0) = 0$, we have $v(t) = 9.8t$. Then the

position is given by $s(t)$ where
$s'(t) = v(t) = 9.8t$, so $s(t) = 4.9t^2 + D$.
Since $s(0) = 0$, we have $s(t) = 4.9t^2$.
Solving $s(t) = 10$ gives $t^2 = \frac{10}{4.9} = \frac{100}{49}$,
so the positive solution is $t = \frac{10}{7}$. The velocity at this time is
$v\left(\frac{10}{7}\right) = 9.8\left(\frac{10}{7}\right) = 14$ m/sec.

(b) Again $v(t) = 9.8t + C$, but this time $v(0) = -2$ and so $v(t) = 9.8t - 2$. Then
$s'(t) = 9.8t - 2$, so $s(t) = 4.9t^2 - 2t + D$.
Since $s(0) = 0$, we have $s(t) = 4.9t^2 - 2t$.
Solving $s(t) = 10$ gives the positive solution $t = \frac{2+10\sqrt{2}}{9.8} \approx 1.647$ sec.
The velocity at this time is
$$v\left(\frac{2+10\sqrt{2}}{9.8}\right) = 9.8\left(\frac{2+10\sqrt{2}}{9.8}\right) - 2 = 10\sqrt{2} \text{ m/sec}$$
or about 14.142 m/sec.

45. Because the function is not continuous on [0, 1]. The function does not satisfy the hypotheses of the Mean Value Theorem, and so it need not satisfy the conclusion of the Mean Value Theorem.

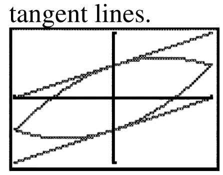
46. Because the Mean Value Theorem applies to the function $y = \sin x$ on any interval, and $y = \cos x$ is the derivative of $\sin x$. So, between any two zeros of $\sin x$, its derivative, $\cos x$, must be zero at least once.

47. $f(x)$ must be zero at least once between a and b by the Intermediate Value Theorem. Now suppose that $f(x)$ is zero twice between a and b. Then by the Mean Value Theorem, $f'(x)$ would have to be zero at least once between the two zeros of $f(x)$, but this can't be true since we are given that $f'(x) \neq 0$ on this interval. Therefore, $f(x)$ is zero once and only once between a and b.

48. Let $f(x) = x^4 + 3x + 1$. Then $f(x)$ is continuous and differentiable everywhere.
$f'(x) = 4x^3 + 3$, which is never zero between $x = -2$ and $x = -1$. Since $f(-2) = 11$ and $f(-1) = -1$, exercise 47 applies, and $f(x)$ has exactly one zero between $x = -2$ and $x = -1$.

49. Let $f(x) = x + \ln(x + 1)$. Then $f(x)$ is continuous and differentiable everywhere on [0, 3]. $f'(x) = 1 + \frac{1}{x+1}$, which is never zero on [0, 3]. Now $f(0) = 0$, so $x = 0$ is one solution of the equation. If there were a second solution, $f(x)$ would be zero twice in [0, 3], and by the Mean Value Theorem, $f'(x)$ would have to be zero somewhere between the two zeros of $f(x)$. But this can't happen, since $f'(x)$ is never zero on [0, 3]. Therefore, $f(x) = 0$ has exactly one solution in the interval [0, 3].

50. Consider the function $k(x) = f(x) - g(x)$. $k(x)$ is continuous and differentiable on $[a, b]$, and since $k(a) = f(a) - g(a) = 0$ and $k(b) = f(b) - g(b) = 0$, by the Mean Value Theorem, there must be a point c in (a, b) where $k'(c) = 0$. But since $k'(c) = f'(c) - g'(c)$, this means that $f'(c) = g'(c)$, and c is a point where the graphs of f and g have parallel or identical tangent lines.



(–1, 1) by [–2, 2]

51. False; for example, the function x^3 is increasing on (–1, 1), but $f'(0) = 0$.

52. True; in fact, f is increasing on $[a, b]$ by Corollary 1 to the Mean Value Theorem.

53. A; $f'(x) = \frac{\frac{1}{2} - 1}{\frac{\pi}{3}} = -\frac{3}{2\pi}$.

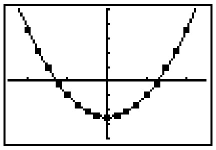
54. B; $f'(x) = e^{x^3-6x^2+8}(3x^2 - 12x) = e^{x^3-6x^2+8}(3x)(x-4)$, which is negative only when x is between 0 and 4.

55. E; $\frac{d}{dx}(2\sqrt{x} - 10) = \frac{2}{2\sqrt{x}} = \frac{1}{\sqrt{x}}$.

56. D; $x^{3/5}$ is not differentiable at $x = 0$.

57. **(a)** Increasing: [–2, –1.3] and [1.3, 2]; decreasing: [–1.3, 1.3]; local max: $x \approx -1.3$ local min: $x \approx 1.3$

(b) Regression equation: $y = 3x^2 - 5$



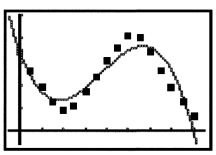
[–2.5, 2.5] by [–8, 10]

(c) Since $f'(x) = 3x^2 - 5$, we have $f(x) = x^3 - 5x + C$. But $f(0) = 0$, so $C = 0$. Then $f(x) = x^3 - 5x$.

58. **(a)** Toward: $0 < t < 2$ and $5 < t < 8$; away: $2 < t < 5$

(b) A local extremum in this problem is a time/place where Priya changes the direction of her motion.

(c) Regression equation:
$y = -0.0820x^3 + 0.9162x^2 - 2.5126x + 3.3779$



[−0.5, 8.5] by [−0.5, 5]

(d) Using the unrounded values from the regression equation, we obtain $f'(t) = -0.2459t^2 + 1.8324t - 2.5126$. According to the regression equation, Priya is moving toward the motion detector when $f'(t) < 0$ $(0 < t < 1.81$ and $5.64 < t < 8)$, and away from the detector when $f'(t) > 0$ $(1.81 < t < 5.64)$.

59. $$\frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{ab}$$

$f'(c) = -\frac{1}{c^2}$, so $-\frac{1}{c^2} = -\frac{1}{ab}$ and $c^2 = ab$.
Thus, $c = \sqrt{ab}$.

60. $$\frac{f(b)-f(a)}{b-a} = \frac{b^2-a^2}{b-a} = b+a$$

$f'(c) = 2c$, so $2c = b + a$ and $c = \frac{a+b}{2}$.

61. By the Mean Value Theorem, $\sin b - \sin a = (\cos c)(b - a)$ for some c between a and b. Taking the absolute value of both sides and using $|\cos c| \le 1$ gives the result.

62. Since differentiability implies continuity, we can apply the Mean Value Theorem to f on $[a, b]$. Since $f(b) < f(a)$, $\frac{f(b)-f(a)}{b-a}$ is negative, and hence $f'(x)$ must be negative at some point between a and b.

63. Let $f(x)$ be a monotonic function defined on an interval D. For any two values in D, we may let x_1 be the smaller value and let x_2 be the larger value, so $x_1 < x_2$. Then either $f(x_1) < f(x_2)$ (if f is increasing), or $f(x_1) > f(x_2)$ (if f is decreasing), which means $f(x_1) \ne f(x_2)$. Therefore, f is one-to-one.

Section 5.3 Connecting f' and f'' with the Graph of f (pp. 209–222)

Exploration 1 Finding f from f'

1. Any function $f(x) = x^4 - 4x^3 + C$ where C is a real number. For example, let $C = 0, 1, 2$. Their graphs are all vertical shifts of each other.

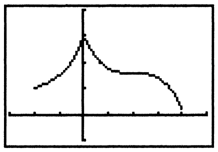
2. Their behavior is the same as the behavior of the function f of Example 8.

Exploration 2 Finding f from f' and f''

1. f has an absolute maximum at $x = 0$ and an absolute minimum of 1 at $x = 4$. We are not given enough information to determine $f(0)$.

2. f has a point of inflection at $x = 2$.

3.



$[-3, 5]$ by $[-5, 20]$

Quick Review 5.3

1. $x^2 - 9 < 0$

$(x+3)(x-3) < 0$

Intervals	$x < -3$	$-3 < x < 3$	$3 < x$
Sign of $(x+3)(x-3)$	+	−	+

Solution set: (–3, 3)

2. $x^3 - 4x > 0$

$x(x+2)(x-2) > 0$

Intervals	$x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x$
Sign of $x(x+2)(x-2)$	−	+	−	+

Solution set: $(-2, 0) \cup (2, \infty)$

3. f: all reals

f': all reals, since $f'(x) = xe^x + e^x$

4. f: all reals

f': $x \neq 0$, since $f'(x) = \dfrac{3}{5}x^{-2/5}$

5. f: $x \neq 2$

f': $x \neq 2$, since $f'(x) = \dfrac{(x-2)(1) - (x)(1)}{(x-2)^2} = \dfrac{-2}{(x-2)^2}$

6. f: all reals

f': $x \neq 0$, since $f'(x) = \dfrac{2}{5}x^{-3/5}$

7. Left end behavior model: 0

Right end behavior model: $-x^2 e^x$

Horizontal asymptote: $y = 0$

8. Left end behavior model: $x^2 e^{-x}$

Right end behavior model: 0

Horizontal asymptote: $y = 0$

9. Left end behavior model: 0
Right end behavior model: 200
Horizontal asymptote: $y = 0$, $y = 200$

10. Left end behavior model: 0
Right end behavior model: 375
Horizontal asymptotes: $y = 0$, $y = 375$

Section 5.3 Exercises

1. $y' = 2x - 1$

Intervals	$x < \frac{1}{2}$	$x > \frac{1}{2}$
Sign of y'	–	+
Behavior of y	Decreasing	Increasing

Local (and absolute) minimum at $\left(\frac{1}{2}, -\frac{5}{4}\right)$

2. $y' = -6x^2 + 12x = -6x(x-2)$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of y'	–	+	–
Behavior of y	Decreasing	Increasing	Decreasing

Local maximum: (2, 5);
local minimum: (0, –3)

3. $y' = 8x^3 - 8x = 8x(x-1)\ (x+1)$

Intervals	$x < -1x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
Sign of y'	–	+	–	+
Behavior of y	Decreasing	Increasing	Decreasing	Increasing

Local maximum: (0, 1); local (and absolute) minima: (–1, –1) and (1, –1)

4. $y' = xe^{1/x}(-x^{-2}) + e^{1/x} = e^{1/x}\left(1 - \frac{1}{x}\right)$

Intervals	$x < 0$	$0 < x < 1$	$1 < x$
Sign of y'	+	–	+
Behavior of y	Increasing	Decreasing	Increasing

Local minimum: (1, e)

5. $y' = x\dfrac{1}{2\sqrt{8-x^2}}(-2x) + \sqrt{8-x^2}(1) = \dfrac{8-2x^2}{\sqrt{8-x^2}}$

Intervals	$-\sqrt{8} < x < -2$	$-2 < x < 2$	$2 < x < \sqrt{8}$
Sign of y'	$-$	$+$	$-$
Behavior of y	Decreasing	Increasing	Decreasing

Local maxima: $\left(-\sqrt{8},\ 0\right)$ and (2, 4);

local minima: (–2, –4) and $\left(\sqrt{8},\ 0\right)$

Note that the local extrema at $x = \pm 2$ are also absolute extrema.

6. $y' = \begin{cases} -2x, & x < 0 \\ 2x, & x > 0 \end{cases}$

Intervals	$x < 0$	$x > 0$
Sign of y'	$+$	$+$
Behavior of y	Increasing	Increasing

Local minimum: (0, 1)

7. $y' = 12x^2 + 42x + 36$

$y'' = 24x + 42 = 6(4x + 7)$

Intervals	$x < -\frac{7}{4}$	$-\frac{7}{4} < x$
Sign of y''	$-$	$+$
Behavior of y	Concave down	Concave up

(a) $\left(-\dfrac{7}{4},\ \infty\right)$

(b) $\left(-\infty,\ -\dfrac{7}{4}\right)$

8. $y' = -4x^3 + 12x^2 - 4$

$y'' = -12x^2 + 24x = -12x(x-2)$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of y''	$-$	$+$	$-$
Behavior of y	Concave down	Concave up	Concave down

(a) (0, 2)

(b) $(-\infty, 0)$ and $(2, \infty)$

9. $y' = \frac{2}{5}x^{-4/5}$

$y'' = -\frac{8}{25}x^{-9/5}$

Intervals	$x < 0$	$0 < x$
Sign of y''	+	−
Behavior of y	Concave up	Concave down

(a) $(-\infty, 0)$

(b) $(0, \infty)$

10. $y' = -\frac{1}{3}x^{-2/3}$

$y'' = \frac{2}{9}x^{-5/3}$

Intervals	$x < 0$	$0 < x$
Sign of y''	−	+
Behavior of y	Concave down	Concave up

(a) $(0, \infty)$

(b) $(-\infty, 0)$

11. $y' = \begin{cases} 2, & x < 1 \\ -2x, & x > 1 \end{cases}$

$y'' = \begin{cases} 0, & x < 1 \\ -2, & x > 1 \end{cases}$

Intervals	$x < 1$	$1 < x$
Sign of y''	0	−
Behavior of y	Linear	Concave down

(a) None

(b) $(1, \infty)$

12. $y' = e^x$

$y'' = e^x$

Since y' and y'' are both positive on the entire domain, y is increasing and concave up on the entire domain.

(a) $(0, 2\pi)$

(b) None

13. $y = xe^x$

$y' = e^x + xe^x$

$y'' = 2e^x + xe^x$

$y'' = 0$ at $x = -2$

Intervals	$x < -2$	$x > -2$
Sign of y''	−	+
Behavior of y	Concave down	Concave up

Inflection point at $\left(-2, -\frac{2}{e^2}\right)$

14. $y = x\sqrt{9 - x^2}$

$y' = \sqrt{9 - x^2} - \frac{x^2}{\sqrt{9 - x^2}}$

$y'' = -\frac{x}{(9 - x^2)^{1/2}} + \frac{x^3 - 18x}{(9 - x^2)^{3/2}} = \frac{x(2x^2 - 27)}{(9 - x^2)^{3/2}}.$

On the domain $[-3, 3]$, $y'' = 0$ only at $x = 0$

Intervals	$-3 < x < 0$	$0 < x < 3$
Sign of y''	+	−
Behavior of y	Concave up	Concave down

Inflection point at (0, 0)

15. $y' = \frac{1}{1 + x^2}$

$$y'' = \frac{d}{dx}(1 + x^2)^{-1}$$
$$= -(1 + x^2)^{-2}(2x)$$
$$= \frac{-2x}{(1 + x^2)^2}$$

Intervals	$x < 0$	$0 < x$
Sign of y''	$+$	$-$
Behavior of y	Concave up	Concave down

Inflection point at (0, 0)

16. $y = x^3(4 - x)$

$y' = 12x^2 - 4x^3$

$y'' = 24x - 12x^2$

Intervals	$x < 0$	$0 < x < 2$	$x > 2$
Sign of y''	$-$	$+$	$-$
Behavior of y	Concave down	Concave up	Concave down

Inflection points at $(0, 0)$ and $(2, 16)$

17. $y = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}$

$y' = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4x - 4}{3x^{2/3}}$

$y'' = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3} = \frac{4x + 8}{9x^{5/3}}$

Intervals	$x < -2$	$-2 < x < 0$	$0 < x$
Sign of y''	$+$	$-$	$+$
Behavior of y	Concave up	Concave down	Concave up

Inflection points at $(-2, 6\sqrt[3]{2}) \approx (-2, 7.56)$ and (0, 0)

18. $y = x^{1/2}(x + 3)$

$y' = \frac{1}{2}x^{-1/2}(x + 3) + x^{1/2}$

$y'' = \frac{1}{(x)^{1/2}} - \frac{x + 3}{4(x)^{3/2}} = 0$

Intervals	$0 < x < 1$	$x > 1$
Sign of y''	$+$	$-$
Behavior of y	Concave up	Concave down

Inflection pt at (1, 4)

19. $y' = \dfrac{(x-2)(3x^2-4x+1)-(x^3-2x^2+x-1)(1)}{(x-2)^2}$

$= \dfrac{2x^3-8x^2+8x-1}{(x-2)^2}$

$y'' = \dfrac{(x-2)^2(6x^2-16x+8)-(2x^3-8x^2+8x-1)(2)(x-2)}{(x-2)^4}$

$= \dfrac{(x-2)(6x^2-16x+8)-2(2x^3-8x^2+8x-1)}{(x-2)^3}$

$= \dfrac{2x^3-12x^2+24x-14}{(x-2)^3}$

$= \dfrac{2(x-1)(x^2-5x+7)}{(x-2)^3}$

Note that the discriminant of x^2-5x+7 is $(-5)^2-4(1)(7)=-3$, so the only solution of $y''=0$ is $x=1$.

Intervals	$x<1$	$1<x<2$	$2<x$
Sign of y''	–	+	–
Behavior of y	Concave up	Concave down	Concave up

Inflection point at (1, 1)

20. $y' = \dfrac{(x^2+1)(1)-x(2x)}{(x^2+1)^2} = \dfrac{-x^2+1}{(x^2+1)^2}$

$y'' = \dfrac{(x^2+1)^2(-2x)-(-x^2+1)(2)(x^2+1)(2x)}{(x^2+1)^4}$

$= \dfrac{(x^2+1)(-2x)-4x(-x^2+1)}{(x^2+1)^3}$

$= \dfrac{2x^3-6x}{(x^2+1)^3} = \dfrac{2x(x^2-3)}{(x^2+1)^3}$

Intervals	$x<-\sqrt{3}$	$-\sqrt{3}<x<0$	$0<x<\sqrt{3}$	$\sqrt{3}<x$
Sign of y'	–	+	–	+
Behavior of y	Concave down	Concave up	Concave down	Concave up

Inflection points at (0, 0), $\left(\sqrt{3}, \dfrac{\sqrt{3}}{4}\right)$, and $\left(-\sqrt{3}, -\dfrac{\sqrt{3}}{4}\right)$

21. **(a)** Zero: $x=\pm 1$;
positive: $(-\infty, -1)$ and $(1, \infty)$;
negative: $(-1, 1)$

(b) Zero: $x = 0$;
positive: $(0, \infty)$;
negative: $(-\infty, 0)$

22. **(a)** Zero: $x \approx 0, \pm 1.25$;
positive: $(-1.25, 0)$ and $(1.25, \infty)$;
negative: $(-\infty, -1.25)$ and $(0, 1.25)$

(b) Zero: $x \approx \pm 0.7$;
positive: $(-\infty, -0.7)$ and $(0.7, \infty)$;
negative: $(-0.7, 0.7)$

23. **(a)** $(-\infty, -2]$ and $[0, 2]$

(b) $[-2, 0]$ and $[2, \infty)$

(c) Local maxima: $x = -2$ and $x = 2$;
local minimum: $x = 0$

24. **(a)** $[-2, 2]$

(b) $(-\infty, -2]$ and $[2, \infty)$

(c) Local maximum: $x = 2$;
local minimum: $x = -2$

25. **(a)** $v(t) = x'(t) = 2t - 4$

(b) $a(t) = v'(t) = 2$

(c) It begins at position 3 moving in a negative direction. It moves to position –1 when $t = 2$, and then changes direction, moving in a positive direction thereafter.

26. **(a)** $v(t) = x'(t) = -2 - 2t$

(b) $a(t) = v'(t) = -2$

(c) It begins at position 6 and moves in the negative direction thereafter.

27. **(a)** $v(t) = x'(t) = 3t^2 - 3$

(b) $a(t) = v'(t) = 6t$

(c) It begins at position 3 moving in a negative direction. It moves to position 1 when $t = 1$, and then changes direction, moving in a positive direction thereafter.

28. **(a)** $v(t) = x'(t) = 6t - 6t^2$

(b) $a(t) = v'(t) = 6 - 12t$

(c) It begins at position 0. It starts moving in the positive direction until it reaches position 1 when $t = 1$, and then it changes direction. It moves in the negative direction thereafter.

29. **(a)** The velocity is zero when the tangent line is horizontal, at approximately $t = 2.2$, $t = 6$ and $t = 9.8$.

(b) The acceleration is zero at the inflection points, approximately $t = 4$, $t = 8$ and $t = 11$.

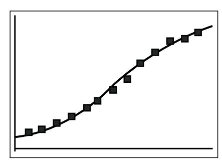
30. **(a)** The velocity is zero when the tangent line is horizontal, at approximately $t = -0.2$, $t = 4$, and $t = 12$.

(b) The acceleration is zero at the inflection points, approximately $t = 1.5$, $t = 5.2$, $t = 8$, $t = 11$, and $t = 13$.

31. Some calculators use different logistic regression equations, so answers may vary.

(a) $y = \dfrac{12655.179}{1 + 12.871e^{-0.0326t}}$

(b)



[0, 140] by [–200, 12000]

(c) $y = \dfrac{12655.179}{1 + 12.871e^{-0.0326(180)}} = 12{,}209{,}870.$
(This is remarkably close to the 2000 census number of 12,281,054.)

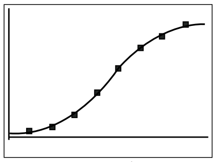
(d) The second derivative has a zero at about 78, indicating that the population was growing fastest in 1898. This corresponds to the inflection point on the regression curve.

(e) The regression equation predicts a population limit of about 12,655,179.

32. Some calculators use different logistic regression equations, so answers may vary.

(a) $y = \dfrac{28984386.288}{1 + 49.252e^{-0.851t}}$

(b)



[0, 9] by [–3.1 × 10^6, 3.2 × 10^7]

(c) The zero of the second derivative is about 4.6, which puts the fastest growth during 1981. This corresponds to the inflection point on the regression curve.

(d) The regression curve predicts that cable subscribers will approach a limit of 28,984,386 + 12,168,450 subscribers (about 41 million).

33. $y = 3x - x^3 + 5$

$y' = 3 - 3x^2$

$y'' = -6x$

$y' = 0$ at ± 1.

$y''(-1) > 0$ and $y''(1) < 0$, so there is a local minimum at (–1, 3) and a local maximum at (1, 7).

34. $y = x^5 - 80x + 100$

$y' = 5x^4 - 80$

$y'' = 20x^3$

$y' = 0$ at ± 2

$y''(-2) < 0$ and $y''(2) > 0$, so there is a local maximum at (–2, 228) and a local minimum at (2, –28).

35. $y = x^3 + 3x^2 - 2$

$y' = 3x^2 + 6x$

$y'' = 6x + 6$

$y' = 0$ at -2 and 0.

$y''(-2) < 0$, $y''(0) > 0$,

$y = x^3 + 3x^2 - 2$

$y = x^3 + 3x^2 - 2$

$y' = 3x^2 + 6x$

$y'' = 6x + 6$

$y' = 0$ at -2 and 0.

$y''(-2) < 0$, $y''(0) > 0$,

so there is a local maximum at (–2, 2) and a local minimum at (0, –2).

36. $y = 3x^5 - 25x^3 + 60x + 20$

$y' = 15x^4 - 75x^2 + 60$

$y'' = 60x^3 - 150x$

$y' = 0$ at ± 1 and ± 2.

$y''(-2) < 0$, $y''(-1) > 0$

$y''(1) < 0$, and $y''(2) > 0$;

so there are local maxima at (–2, 4) and (1, 58), and there are local minima at (–1, –18) and (2, 36).

37. $y = xe^x$

$y' = (x+1)e^x$

$y'' = (x+2)e^x$

$y' = 0$ at -1.

$y''(-1) > 0$, so there is a local minimum at $\left(-1, -\frac{1}{e}\right)$.

38. $y = xe^{-x}$

$y' = (1-x)e^{-x}$

$y'' = (x-2)e^{-x}$

$y' = 0$ at 1

$y''(1) < 0$, so there is a local maximum at $\left(1, \frac{1}{e}\right)$.

39. $y' = (x-1)^2(x-2)$

Intervals	$x < 1$	$1 < x < 2$	$2 < x$
Sign of y'	–	–	+
Behavior of y	Decreasing	Decreasing	Increasing

$$\begin{aligned} y'' &= (x-1)^2(1) + (x-2)(2)(x-1) \\ &= (x-1)[(x-1) + 2(x-2)] \\ &= (x-1)(3x-5) \end{aligned}$$

Intervals	$x < 1$	$1 < x < \frac{5}{3}$	$\frac{5}{3} < x$
Sign of y''	+	–	+
Behavior of y	Concave up	Concave down	Concave up

(a) There are no local maxima.

(b) There is a local (and absolute) minimum at $x = 2$.

(c) There are points of inflection at $x = 1$ and at $x = \frac{5}{3}$.

40. $y' = (x-1) \cdot (x-2)(x-4)$

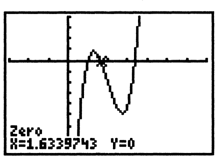
Intervals	$x < 1$	$1 < x < 2$	$2 < x < 4$	$4 < x$
Sign of y'	+	+	–	+
Behavior of y	Increasing	Increasing	Decreasing	Increasing

$$\begin{aligned} y'' &= \frac{d}{dx}[(x-1)^2(x^2-6x+8)] \\ &= (x-1)^2(2x-6)+(x^2-6x+8)(2)(x-1) \\ &= (x-1)[(x-1)(2x-6)+2(x^2-6x+8)] \\ &= (x-1)(4x^2-20x+22) \\ &= 2(x-1)(2x^2-10x+11) \end{aligned}$$

Note that the zeros of y'' are $x = 1$ and

$$\begin{aligned} x &= \frac{10 \pm \sqrt{10^2 - 4(2)(11)}}{4} \\ &= \frac{10 \pm \sqrt{12}}{4} \\ &= \frac{5 \pm \sqrt{3}}{2} \approx 1.63 \text{ or } 3.37. \end{aligned}$$

The zeros of y'' can also be found graphically, as shown.



[−3, 7] by [−8, 4]

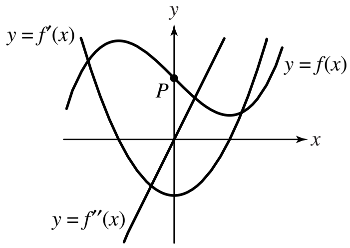
Intervals	$x < 1$	$1 < x < 1.63$	$1.63 < x < 3.37$	$3.37 < x$
Sign of y''	–	+	–	+
Behavior of y	Concave down	Concave up	Concave down	Concave up

(a) Local maximum at $x = 2$

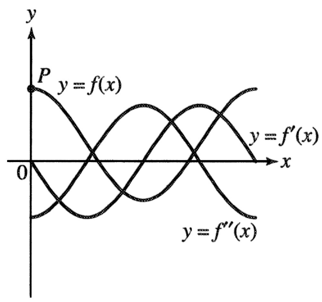
(b) Local minimum at $x = 4$

(c) Points of inflection at $x = 1$, at $x \approx 1.63$, and at $x \approx 3.37$.

41.



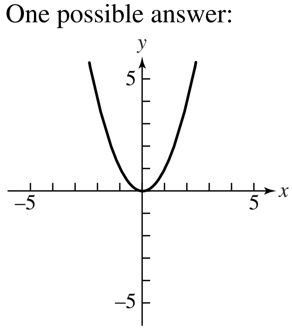
42.



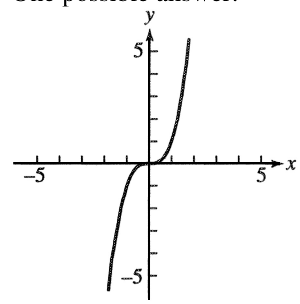
43. No. f must have a horizontal tangent at that point, but f could be increasing (or decreasing), and there would be no local extremum. For example, if $f(x) = x^3$, $f'(0) = 0$ but there is no local extremum at $x = 0$.

44. No; $f''(x)$ could still be positive (or negative) on both sides of $x = c$, in which case the concavity of the function would not change at $x = c$. For example, if $f(x) = x^4$, then $f''(0) = 0$, but f has no inflection point at $x = 0$.

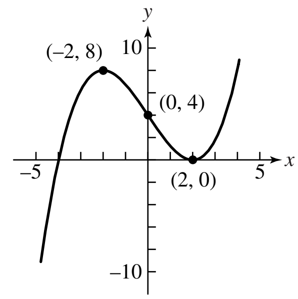
45. One possible answer:



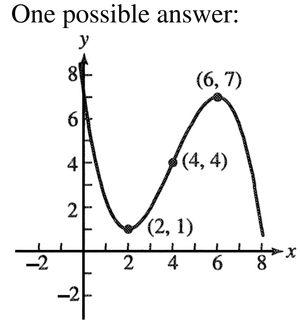
46. One possible answer:



47. One possible answer:



48. One possible answer:



49. **(a)** [0, 1], [3, 4], and [5.5, 6]

(b) [1, 3] and [4, 5.5]

(c) Local maxima: $x = 1$, $x = 4$ (if f is continuous at $x = 4$), and $x = 6$; local minima: $x = 0$, $x = 3$, and $x = 5.5$

50. If f is continuous on the interval [0, 3]:

(a) [0, 3]

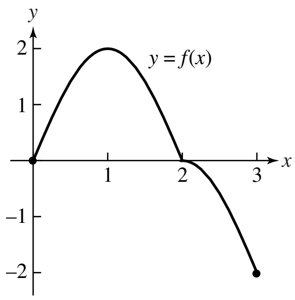
(b) Nowhere

(c) Local maximum: $x = 3$; local minimum: $x = 0$

51. **(a)** Absolute maximum at (1, 2); absolute minimum at (3, –2)

(b) None

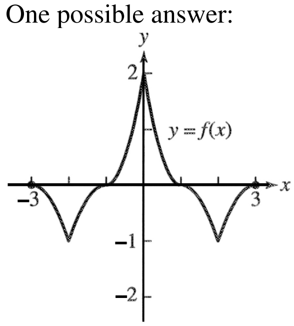
(c) One possible answer:



52. **(a)** Absolute maximum at (0, 2); absolute minimum at (2, –1) and (–2, –1)

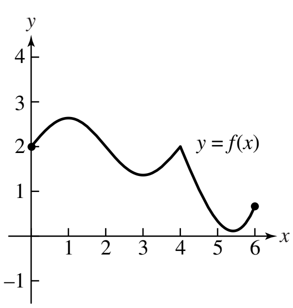
(b) At (1, 0) and (–1, 0)

(c) One possible answer:

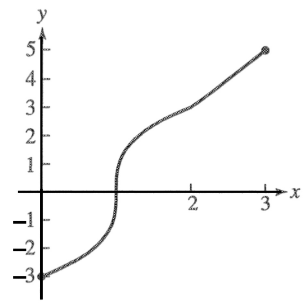


(d) Since f is even, we know $f(3) = f(-3)$. By the continuity of f, since $f(x) < 0$ when $2 < x < 3$, we know that $f(3) \le 0$, and since $f(2) = -1$ and $f'(x) > 0$ when $2 < x < 3$, we know that $f(3) > -1$. In summary, we know that $f(3) = f(-3)$, $-1 < f(3) \le 0$, and $-1 < f(-3) \le 0$.

53.



54.



55. False. For example, consider $f(x) = x^4$ at $c = 0$.

56. True. This is the Second Derivative Test for a local maximum.

57. A; $y = ax^3 + 3x^2 + 4x + 5$ say $a = -2$

$y' = -6x^2 + 6x + 4$
$y'' = -12x + 6$
$y'' = 0$ at $\frac{1}{2}$

Intervals	$x < \frac{1}{2}$	$x > \frac{1}{2}$
Sign of y''	+	–
Behavior of y	Concave up	Concave down

58. E

59. C; $y = x^5 - 5x^4 + 3x + 7$

$y' = 5x^4 - 20x^3 + 3$
$y'' = 20x^3 - 60x^2 = 20x^2(x-3)$

Note that $y'' = 0$ at $x = 0$ and $x = 3$, but y'' only changes sign at $x = 3$.

Intervals	$x < 3$	$x > 3$
Sign of y''	–	+
Behavior of y	Concave down	Concave up

(3, –146) is an inflection point.

60. A. There is a local maximum of f' at $x = c$.

61. **(a)** In exercise 7, $a = 4$ and $b = 21$, so $-\frac{b}{3a} = -\frac{7}{4}$, which is the x-value where the point of inflection occurs.

The local extrema are at $x = -2$ and $x = -\frac{3}{2}$, which are symmetric about $x = -\frac{7}{4}$.

(b) In exercise 2, $a = -2$ and $b = 6$, so $-\frac{b}{3a} = 1$, which is the x-value where the point of inflection occurs.

The local extrema are at $x = 0$ and $x = 2$, which are symmetric about $x = 1$.

(c) $f'(x) = 3ax^2 + 2bx + c$ and
$f''(x) = 6ax + 2b$.
The point of inflection will occur where
$f''(x) = 0$, which is at $x = -\frac{b}{3a}$.
If there are local extrema, they will occur at the zeros of $f'(x)$. Since $f'(x)$ is quadratic, its graph is a parabola and any zeros will be symmetric about the vertex which will also be where $f''(x) = 0$.

62. **(a)**
$$f'(x) = \frac{(1+ae^{-bx})(0)-(c)(-abe^{-bx})}{(1+ae^{-bx})^2}$$
$$= \frac{abce^{-bx}}{(1+ae^{-bx})^2}$$
$$= \frac{abce^{+bx}}{(e^{bx}+a)^2},$$
so the sign of $f'(x)$ is the same as the sign of abc.

(b)
$$f''(x) = \frac{(e^{bx}+a)^2(ab^2ce^{bx})-(abce^{bx})2(e^{bx}+a)(be^{bx})}{(e^{bx}+a)^4}$$
$$= \frac{(e^{bx}+a)(ab^2ce^{bx})-(abce^{bx})(2be^{bx})}{(e^{bx}+a)^3}$$
$$= -\frac{ab^2ce^{bx}(e^{bx}-a)}{(e^{bx}+a)^3}$$
Since $a > 0$, this changes sign when $x = \frac{\ln a}{b}$ due to the $e^{bx} - a$ factor in the numerator, and $f(x)$ has a point of inflection at the location.

63. **(a)** $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$
$f''(x) = 12ax^2 + 6bx + 2c$
Since $f''(x)$ is quadratic, it must have 0, 1, or 2 zeros. A quadratic with 0 or 1 zeros never changes sign, so *f* has no points of infection if $f''(x)$ has 0 or 1 zeros. If $f''(x)$ has 2 zeros, it will change sign twice, and $f(x)$ will have 2 points of inflection.

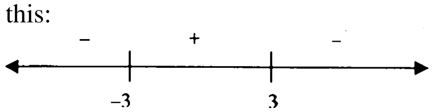
(b) $f(x)$ has two points of inflection if and only if $3b^2 > 8ac$.

Quick Quiz Sections 5.1–5.3

1. (C)
$$f'(x) = 5(x-2)^4(x+3)^4 + 4(x-2)^5(x+3)^3 = 0$$
$$x = -3, -\frac{7}{9}, 2$$

2. (D) $f'(x) = (x-3)^2 + 2(x-2)(x-3) = (x-3)(3x-7)$
$f'(x) = 0$ when $x = 3$ or $x = \frac{7}{3}$
$$f''(x) = 6x - 16$$
$$f''(3) = 2 > 0$$
$$f''\left(\frac{7}{3}\right) = -2 < 0$$
Relative maximum is at $x = \frac{7}{3}$ only.

3. (B) $x^2 - 9 = 0$
$x = \pm 3$
$$f''(x) = 2x$$
$$f''(3) = 6 > 0$$
$$f''(-3) = -6 < 0$$
$f'(x) = (x^2 - 9)g(x)$; where $g(x) < 0$ for all x. Thus the sign graph for $f'(x)$ looks like this:



By the First Derivative Test, f has a relative maximum at $x = -3$ and a relative minimum at $x = 3$.

4. **(a)** $\frac{d}{dx}\left(3\ln(x^2+2) - 2x\right) = 3\frac{2x}{x^2+2} - 2 = 0$
$x = 1, 2$

Intervals	$-2 < x < 1$	$1 < x < 2$	$2 < x < 4$
Sign of y'	–	+	–
Behavior of y	Decreasing	Increasing	Decreasing

f has relative minima at $x = 1$ and $x = 4$, f has relative maxima at $x = \pm 2$

(b) $f''(x) = \frac{d}{dx}\left(\frac{6x}{x^2+2} - 2\right)$
$$f''(x) = \frac{6}{x^2+2} - \frac{12x^2}{(x^2+2)^2} = 0$$
$$x = \pm\sqrt{2}$$
f has points of inflection at $x = \pm\sqrt{2}$

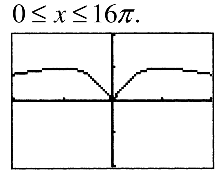
(c) The absolute maximum is at $x = -2$ and $f(x) = 3\ln 6 + 4$.

Section 5.4 Modeling and Optimization (pp. 223–236)

Exploration 1 Constructing Cones

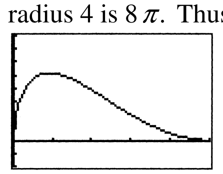
1. The circumference of the base of the cone is the circumference of the circle of radius 4 minus x, or $8\pi - x$. Thus, $r = \frac{8\pi - x}{2\pi}$. Use the Pythagorean Theorem to find h, and the formula for the volume of a cone to find V.

2. The expression under the radical must be nonnegative, that is, $16 - \left(\frac{8\pi - x}{2\pi}\right)^2 \geq 0$.
Solving this inequality for x gives: $0 \leq x \leq 16\pi$.



[0, 16π] by [−10, 40]

3. The circumference of the original circle of radius 4 is 8π. Thus, $0 \leq x \leq 8\pi$.



[0, 8π] by [−10, 40]

4. The maximum occurs at about $x = 4.61$. The maximum volume is about $V = 25.80$.

5. Start with $\frac{dV}{dx} = \frac{2\pi}{3}rh\frac{dr}{dx} + \frac{\pi}{3}r^2\frac{dh}{dx}$.
Compute $\frac{dr}{dx}$ and $\frac{dh}{dx}$, substitute these values in $\frac{dV}{dx}$, set $\frac{dV}{dx} = 0$, and solve for x to obtain

$x = \dfrac{8(3-\sqrt{6})\pi}{3} \approx 4.61$. Then

$V = \dfrac{128\pi\sqrt{3}}{27} \approx 25.80$.

Quick Review 5.4

1. $y' = 3x^2 - 12x + 12 = 3(x-2)^2$
   Since $y' \geq 0$ for all x (and is increasing on $y' > 0$ for $x \neq 2$), y is increasing on $(-\infty, \infty)$ and there are no local extrema.

2. $y' = 6x^2 + 6x - 12 = 6(x+2)(x-1)$
   $y'' = 12x + 6$
   The critical points occur at $x = -2$ or $x = 1$, since $y' = 0$ at these points. Since $y''(-2) = -18 < 0$, the graph has a local maximum at $x = -2$. Since $y''(1) = 18 > 0$, the graph has a local minimum at $x = 1$. In summary, there is a local maximum at $(-2, 17)$ and a local minimum at $(1, -10)$.

3. $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi(5)^2(8) = \dfrac{200\pi}{3}\text{cm}^3$

4. $V = \pi r^2 h = 1000$
   $SA = 2\pi rh + 2\pi r^2 = 600$
   Solving the volume equation for h gives $= \dfrac{1000}{\pi r^2}$. Substituting into the surface area equation gives $\dfrac{2000}{r} + 2\pi r^2 = 600$. Solving graphically, we have $r \approx -11.14$, $r \approx 4.01$, or $r \approx 7.13$. Discarding the negative value and using $h = \dfrac{1000}{\pi r^2}$ to find the corresponding values of h, the two possibilities for the dimensions of the cylinder are:
   $r \approx 4.01$ cm and $h \approx 19.82$ cm, or,
   $r \approx 7.13$ cm and $h \approx 6.26$ cm.

5. Since $y = \sin x$ is an odd function, $\sin(-\alpha) = -\sin\alpha$.

6. Since $y = \cos x$ is an even function, $\cos(-\alpha) = \cos\alpha$.

7. $$\begin{aligned}\sin(\pi-\alpha) &= \sin\pi\cos\alpha - \cos\pi\sin\alpha \\ &= 0\cos\alpha - (-1)\sin\alpha \\ &= \sin\alpha\end{aligned}$$

8. $$\begin{aligned}\cos(\pi-\alpha) &= \cos\pi\cos\alpha + \sin\pi\sin\alpha \\ &= (-1)\cos\alpha + 0\sin\alpha \\ &= -\cos\alpha\end{aligned}$$

9. $x^2 + y^2 = 4$ and $y = \sqrt{3}x$
   $$\begin{aligned}x^2 + (\sqrt{3}x)^2 &= 4 \\ x^2 + 3x^2 &= 4 \\ 4x^2 &= 4 \\ x &= \pm 1\end{aligned}$$
   Since $y = \sqrt{3}x$, the solutions are: $x = 1$ and $y = \sqrt{3}$, or, $x = -1$ and $y = -\sqrt{3}$.
   In ordered pair notation, the solutions are $(1, \sqrt{3})$ and $(-1, -\sqrt{3})$.

10. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ and $y = x + 3$
    $$\begin{aligned}\frac{x^2}{4} + \frac{(x+3)^2}{9} &= 1 \\ 9x^2 + 4(x+3)^2 &= 36 \\ 9x^2 + 4x^2 + 24x + 36 &= 36 \\ 13x^2 + 24x &= 0 \\ x(13x+24) &= 0\end{aligned}$$
    $x = 0$ or $x = -\dfrac{24}{13}$
    Since $y = x + 3$, the solutions are:
    $x = 0$ and $y = 3$, or, $x = -\dfrac{24}{13}$ and $y = \dfrac{15}{13}$.
    In ordered pair notation, the solution are $(0, 3)$ and $\left(-\dfrac{24}{13}, \dfrac{15}{13}\right)$.

Section 5.4 Exercises

1. Represent the numbers by x and $20 - x$, where $0 \leq x \leq 20$.

   (a) The sum of the squares is given by $f(x) = x^2 + (20-x)^2 = 2x^2 - 40x + 400$. Then $f'(x) = 4x - 40$. The critical point and endpoints occur at $x = 0$, $x = 10$, and $x = 20$. Then $f(0) = 400$, $f(10) = 200$, and $f(20) = 400$. The sum of the squares is as large as possible for the numbers 0 and 20, and is as small as possible for the numbers 10 and 10.

(b) The sum of one number plus the square root of the other is given by $g(x) = x + \sqrt{20-x}$. Then $g'(x) = 1 - \frac{1}{2\sqrt{20-x}}$. The critical point occurs when $2\sqrt{20-x} = 1$, so $20 - x = \frac{1}{4}$ and $x = \frac{79}{4}$. Testing the endpoints and critical point, we find $g(0) = \sqrt{20} \approx 4.47$, $g\left(\frac{79}{4}\right) = \frac{81}{4} = 20.25$, and $g(20) = 20$. The sum $x + \sqrt{20-x}$ is as large as possible when the numbers are $x = \frac{79}{4}$ and $20 - x = \frac{1}{4}$. The sum $x + \sqrt{20-x}$ is as small as possible when the numbers are $x = 0$ and $20 - x = 20$.

2. Let x and y represent the legs of the triangle, and note that $0 < x < 5$. Then $x^2 + y^2 = 25$, so $y = \sqrt{25-x^2}$ (since $y > 0$). The area is $A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{25-x^2}$, so

$$\frac{dA}{dx} = \frac{1}{2}x\frac{1}{2\sqrt{25-x^2}}(-2x) + \frac{1}{2}\sqrt{25-x^2}$$
$$= \frac{25-2x^2}{2\sqrt{25-x^2}}.$$

The critical point occurs when $25 - 2x^2 = 0$, which means $x = \frac{5}{\sqrt{2}}$, (since $x > 0$). This value corresponds to the largest possible area, since $\frac{dA}{dx} > 0$ for $0 < x < \frac{5}{\sqrt{2}}$ and $\frac{dA}{dx} < 0$ for $\frac{5}{\sqrt{2}} < x < 5$. When $x = \frac{5}{\sqrt{2}}$, we have $y = \sqrt{25 - \left(\frac{5}{\sqrt{2}}\right)^2} = \frac{5}{\sqrt{2}}$ and $A = \frac{1}{2}xy = \frac{1}{2}\left(\frac{5}{\sqrt{2}}\right)^2 = \frac{25}{4}$. Thus, the largest possible area is $\frac{25}{4}\text{cm}^2$, and the dimensions (legs) are $\frac{5}{\sqrt{2}}$cm by $\frac{5}{\sqrt{2}}$cm.

3. Let x represent the length of the rectangle in inches ($x > 0$).
Then the width is $\frac{16}{x}$ and the perimeter is $P(x) = 2\left(x + \frac{16}{x}\right) = 2x + \frac{32}{x}$.
Since $P'(x) = 2 - 32x^{-2} = \frac{2(x^2-16)}{x^2}$ this critical point occurs at $x = 4$. Since $P'(x) < 0$ for $0 < x < 4$ and $P'(x) > 0$ for $x > 4$, this critical point corresponds to the minimum perimeter. The smallest possible perimeter is $P(4) = 16$ in., and the rectangle's dimensions are 4 in. by 4 in.

4. Let x represent the length of the rectangle in meters ($0 < x < 4$). Then the width is $4 - x$ and the area is $A(x) = x(4 - x) = 4x - x^2$. Since $A'(x) = 4 - 2x$, the critical point occurs at $x = 2$. Since $A'(x) > 0$ for $0 < x < 2$ and $A'(x) < 0$ for $2 < x < 4$, this critical point corresponds to the maximum area. The rectangle with the largest area measures 2 m by 4 – 2 = 2m, so it is a square.

5. **(a)** The equation of line AB is $y = -x + 1$, so the y-coordinate of P is $-x + 1$.

(b) $A(x) = 2x(1 - x)$

(c) Since $A'(x) = \frac{d}{dx}(2x - 2x^2) = 2 - 4x$, the critical point occurs at $x = \frac{1}{2}$. Since $A'(x) > 0$ for $0 < x < \frac{1}{2}$ and $A'(x) < 0$ for $\frac{1}{2} < x < 1$, this critical point corresponds to the maximum area. The largest possible area is $A\left(\frac{1}{2}\right) = \frac{1}{2}$ square unit, and the dimensions of the rectangle are $\frac{1}{2}$unit by 1 unit.

6. If the upper right vertex of the rectangle is located at $(x, 12 - x^2)$ for $0 < x < \sqrt{12}$, then the rectangle's dimensions are $2x$ by $12 - x^2$ and the area is A is

$(x) = 2x\,(12 - x^2) = 24x - 2\,x^3$. Then $A'(x) = 24 - 6x^2 = 6(4 - x^2)$, so the critical point $\left(\text{for } 0 < x < \sqrt{12}\right)$ occurs at $x = 2$. Since $A'(x) > 0$ for $0 < x < 2$ and $A'(x) < 0$ for $2 < x < \sqrt{12}$, this critical point corresponds to the maximum area. The largest possible area is $A(2) = 32$, and the dimensions are 4 by 8.

7. Let x be the side length of the cut-out square $(0 < x < 4)$. Then the base measures $8 - 2x$ in. by $15 - 2x$ in., and the volume is

$$V(x) = x(8-2x)(15-2x)$$
$$= 4x^3 - 46x^2 + 120x.$$

Then

$$V'(x) = 12x^2 - 92x + 120 = 4(3x-5)(x-6).$$

Then the critical point (in $0 < x < 4$) occurs at $x = \frac{5}{3}$. Since $V'(x) > 0$ for $0 < x < \frac{5}{3}$ and $V'(x) < 0$ for $\frac{5}{3} < x < 4$, the critical point corresponds to the maximum volume. The maximum volume is $V\left(\frac{5}{3}\right) = \frac{2450}{27} \approx 90.74\,\text{in}^3$, and the dimensions are $\frac{5}{3}$ in. by $\frac{14}{3}$ in. by $\frac{35}{3}$ in.

8. Note that the values a and b must satisfy $a^2 + b^2 = 20^2$ and so $b = \sqrt{400 - a^2}$. Then the area is given by $A = \frac{1}{2}ab = \frac{1}{2}a\sqrt{400 - a^2}$ for $0 < a < 20$, and

$$\frac{dA}{da} = \frac{1}{2}a\left(\frac{1}{2\sqrt{400-a^2}}\right)(-2a) + \frac{1}{2}\sqrt{400-a^2}$$
$$= \frac{-a^2 + (400 - a^2)}{2\sqrt{400-a^2}}$$
$$= \frac{200 - a^2}{\sqrt{400-a^2}}.$$

The critical point occurs when $a^2 = 200$. Since $\frac{dA}{da} > 0$ for $0 < a < \sqrt{200}$ and $\frac{dA}{da} < 0$ for $\sqrt{200} < a < 20$, this critical point corresponds to the maximum area. Furthermore, if $a = \sqrt{200}$ then $b = \sqrt{400 - a^2} = \sqrt{200}$, so the maximum area occurs when $a = b$.

9. Let x be the length in meters of each side that adjoins the river. Then the side parallel to the river measures $800 - 2x$ meters and the area is $A(x) = x(800 - 2x) = 800x - 2x^2$ for $0 < x < 400$. Therefore, $A'(x) = 800 - 4x$ and the critical point occurs at $x = 200$. Since $A'(x) > 0$ for $0 < x < 200$ and $A'(x) < 0$ for $200 < x < 400$, the critical point corresponds to the maximum area. The largest possible area is $A(200) = 80{,}000\ \text{m}^2$ and the dimensions are 200 m (perpendicular to the river) by 400 m (parallel to the river).

10. If the subdividing fence measures x meters, then the pea patch measures x m by $\frac{216}{x}$ m and the amount of fence needed is $f(x) = 3x + 2\left(\frac{216}{x}\right) = 3x + 432x^{-1}$. Then $f'(x) = 3 - 432x^{-2}$ and the critical point (for $x > 0$) occurs at $x = 12$. Since $f'(x) < 0$ for $0 < x < 12$ and $f'(x) > 0$ for $x > 12$, the critical point corresponds to the minimum total length of fence. The pea patch will measure 12 m by 18 m (with a 12-m divider), and the total amount of fence needed is $f(12) = 72\,\text{m}$.

11. **(a)** Let x be the length in feet of each side of the square base. Then the height is $\frac{500}{x^2}$ ft and the surface area (not including the open top) is

$$S(x) = x^2 + 4x\left(\frac{500}{x^2}\right) = x^2 + 2000x^{-1}.$$

Therefore,

$$S'(x) = 2x - 2000x^{-2} = \frac{2(x^3 - 1000)}{x^2}$$ and

the critical point occurs at $x = 10$. Since $S'(x) < 0$ for $0 < x < 10$ and $S'(x) > 0$ for $x > 10$, the critical point corresponds to the minimum amount of steel used. The dimensions should be 10 ft by 10 ft by 5 ft, where the height is 5 ft.

(b) Assume that the weight is minimized when the total area of the bottom and the four sides is minimized.